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SECTION 14

MINIMUM DISTURBANCE EFFECTS CONTROL

OF LINEAR SYSTEMS WITH LINEAR CONTROLLERS

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ROCKET BOOSTER CONTROL, SECTION 14:

MINIMUM DISTURBANCE EFFECTS CONTROL

OF LINEAR SYSTEMS WITH LINEAR CONTROLLERS

☒ OTS

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(NASA Contract NASw-563)

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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MINIMUM DISTURBANCE EFFECTS CONTROL OF LINEAR SYSTEMS WITH LINEAR CONTROLLERS*

By C. A. Harvey‡

16761

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ABSTRACT

An optimal control problem for linear systems is considered. The optimal controller is one from a class of allowable controllers that minimizes the effect of the worst possible disturbance from the class of uniformly bounded disturbances. When the class of allowable controllers is a family of fixed-gain controllers a method of solution is presented. For sufficiently simple systems the optimal controller may be found by analytical techniques. However, for most realistic systems the use of a computer is required to determine the optimal controller.

As examples, two second order systems are discussed with the optimal controllers being obtained analytically, and a fourth order system is discussed which corresponds to the rigid body representation of a launch booster. This example required the use of a computer to obtain an approximation to the optimal controller. Some discussion of the computation time and computer results is given.

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INTRODUCTION

An optimal control problem is considered for a system represented by the vector differential equation

$$\dot{x} = Ax + bu(x,g) + cg, \quad x(0) = 0 \tag{1}$$

where: x is an n-vector describing the state of the system,

$u(x,g)$ is a scalar-valued control variable,

g is a scalar disturbance,

A is a constant nxn matrix, and

b and c are constant n-vectors.

A fixed time interval $[0,T]$, is assumed to be of interest. Classes of allowable controllers, U, and admissible disturbances, G, are assumed and the control criterion is defined to be

$$C(u) = \max_{1 \leq i \leq m} C_i(u), \qquad C_i(u) = \max_{0 \leq t \leq T} \ \max_{g \in G} \ |d^i \cdot x(t;u,g)| \tag{2}$$

for each u in the class U, with $x(t;u,g)$ denoting the solution of (1), and d^i represents a nonzero constant weighting vector for $i = 1,2,\ldots,m$ and m is any positive integer. An optimal controller is defined as an element of U that minimizes C(u) given by (2), i.e., u_o is optimal if u_o is an element of U and $C(u_o) \leq C(u)$ for every u in the class U.

The class U is assumed to be a class of linear fixed-gain controllers and G is taken to be the class of all measurable functions uniformly bounded in absolute value by one. A method of computing C(u) for an arbitrary u in the class U is developed. This computation can be accomplished by integrating a system of n+m first-order differential equations. These are autonomous ordinary differential equations, n of which are linear and the

remainder being piece-wise linear. This rather simple method of computing C(u) is obtained by using results from the theory of time optimal control of linear systems.

For sufficiently simple problems it is possible to perform the necessary integrations analytically to obtain C(u) explicitly as a function of the gains. Then it may be possible to analytically solve for the minimum of the function C(u) as a function of the gains. If it is not possible to do this analytically, numerical techniques could be used.

When it is not possible to obtain C(u) explicitly as a function of the gains, it is not difficult to compute C(u) numerically. Then iterative or sequential techniques could be used to obtain an approximation to the optimal controller.

Examples are presented to illustrate these methods. Two second-order examples are given followed by a fourth order example which corresponds to the rigid body representation of a launch vehicle.

DEVELOPMENT OF THE COMPUTATIONAL METHOD

Before proceeding, certain notation will be introduced. For the problem stated in the introduction u(x,g) may be written as

$$u(x,g) = \sum_{i=1}^{n} k_i x_i + k_{n+1} g \tag{3}$$

Let k denote the n-vector with components k_i, $i = 1,2,\ldots,n$. Let k* denote the n+1-vector with components k_i, $i = 1,2,\ldots,n+1$. There is a one-to-one correspondence between u(x,g) and k* so that the allowable class U may be specified as an allowable class

of vectors K^*. Dependence on u will be replaced by dependence on k^*. Thus $x(t;\ k^*,g)$ will denote the solution of (1) and $C(k^*)$ and $C_1(k^*)$ will be used instead of $C(u)$ and $C_1(u)$ in (2) when $u(x,g)$ is given by (3). If $u(x,g)$ from (3) is substituted into (1) the result may be written as

$$\dot{x} = (A + bk')x + (c + bk_{n+1})g, \quad x(0) = 0 \tag{4}$$

with prime denoting transpose. The solution of (4) is $x(t;k^*,g)$.

For any choice of k^* and any value of t the set of attainability, $K(t;k^*)$, and the cone of attainability, $C(t;k^*)$, are defined as

$$K(t;k^*) = \left\{\xi : \xi = x(t;k^*,g),\ g \epsilon G\right\} \tag{5}$$

$$C(t;k^*) = \bigcup_{s \epsilon [0,t]} K(s;k^*) \tag{6}$$

These concepts are quite widely discussed in the literature concerned with time optimal control theory. It is possible to express $C_1(k^*)$ as

$$C_1(k^*) = \max_{\xi \epsilon C(T;k^*)} |d^1 \cdot \xi| \tag{7}$$

The expression for $C_1(k^*)$ in (7) is just a restatement of the expression in (2) in terms of the cone of attainability. However, this makes possible the use of known properties of the cone and set of attainability for the system (4) with the assumed class of disturbances, G. It will be shown that $C(T;k^*)=K(T;k^*)$. Then known properties of $K(T;k^*)$ will be cited which lead to a rather complicated but easily computed expression for $C_1(k^*)$.

To establish that $C(T;k^*) = K(T;k^*)$, it will be shown that

$K(s;k^*)$ is contained in $K(T;k^*)$ if $0 \leq s \leq T$. Let s be chosen in $[0,T]$ and let ξ be any point in $K(s;k^*)$. Then there is a g_ξ in G such that

$$\xi = x(s;k^*,g_\xi).$$

Let a shifted disturbance g be defined in terms of g_ξ by

$$g(t) = \begin{cases} 0, & 0 \leq t < T-s \\ g_\xi(t-T+s), & T-s \leq t \leq T \end{cases} \tag{8}$$

Then the point corresponding to the solution to (4) at T with g given by (8) may be written as

$$x(T;k^*,g) = \int_0^T \exp[(A + bk')\,(T-t)]\,(c + bk_{n+1})\,g(t)\,dt$$

$$= \int_{T-s}^T \exp[(A + bk')\,(T-t)]\,(c + bk_{n+1})\,g_\xi(t-T+s)\,dt$$

$$= \int_0^s \exp[(A + bk')\,(s-v)]\,(c + bk_{n+1})\,g_\xi(v)\,dv$$

$$= x(s;k^*,g_\xi)$$

Hence ξ is in $K(T;k^*)$ which establishes that $K(s;k^*) \subset K(T;k^*)$. It is clear from this result and the definition of $C(T;k^*)$ that $K(T;k^*) = C(T;k^*)$.

Replacing $C(T;k^*)$ by $K(T;k^*)$ in (7) the expression for $C_i(k^*)$ becomes

$$C_i(k^*) = \max_{\xi \in K(T;k^*)} |d^i \cdot \xi| \tag{9}$$

This is equivalent to the following expression for $C_i(u)$ corresponding to the notation in (2):

$$C_i(u) = \max_{g \in G} |d^i \cdot x(T;u,g)|.$$

A class of extremal disturbances will now be defined. For any $\eta^o \neq 0$ let $\eta(t;k,\eta^o)$ be defined as $\eta(t;k,\eta^o) = \exp[-(A+bk')'t]\eta^o$ and let $g(t;k^*,\eta^o) = \text{sgn}[\eta(t;k,\eta^o) \cdot (c + bk_{n+1})]$, $0 \leq t \leq T$. Such disturbances, $g(t;k^*,\eta^o)$, will be called extremal disturbances because of the relation between these disturbances and boundary points of $K(T;k^*)$.

It is known that $K(T;k^*)$ is compact and convex. Also, for each $\eta^o \neq 0$ the following inequality has been established (reference 1).

$$\eta(T;k,\eta^o) \cdot x(T;k^*,g) \geq \eta(T;k,\eta^o) \cdot \xi \tag{10}$$

for all $\xi \epsilon K(T;k^*)$ when $g = g(t;k^*,\eta^o)$, $0 \leq t \leq T$.

Another property of $K(T;k^*)$ which is easily shown is that $K(T;k^*)$ is symmetric with respect to the origin, i.e., if $\xi \epsilon K(T;k^*)$ then $(-\xi) \epsilon K(T;k^*)$. This follows because $g \epsilon G$ implies $(-g) \epsilon G$ and $x(T;k^*,-g) = -x(T;k^*,g)$. Thus the absolute value sign may be removed from (9) which gives,

$$C_i(k^*) = \max_{\xi \epsilon K(T;k^*)} d^i \cdot \xi \tag{11}$$

From (10) it is clear that

$$\max_{\xi \epsilon K(T;k^*)} d^i \cdot \xi = d^i \cdot x(T;k^*,g^i)$$

with $g^i = g(t;k^*,\eta^o(i,k))$, $0 \leq t \leq T$, where $\eta^o(i,k)$ is defined by

$$\eta^o(i,k) = \exp\,[A + bk')'T]\; d^i,$$

that is, $\eta(T;k,\eta^o(i,k)) = d^i$. With these definitions of g^i and $\eta^o(i,k)$ it is possible to write

$$C_i(k^*) = d^i \cdot x(T;k^*,g^i) \tag{12}$$

It would not be too difficult to compute $C_i(k^*)$ by first computing $\eta^o(i,k)$, then g^i and $x(T;k^*,g^i)$. The value of $\eta^o(i,k)$ is given by the value (at time T) of the solution to the system

$$\dot{y} = (A + bk')'\,y,\; y(0) = d^i \tag{13}$$

Then $\eta(t;k,\eta^o)$ is the solution of

$$\dot{z} = -(A + bk')'\,z, \quad z(0) = \eta^o \tag{14}$$

and

$$g^i = \operatorname{sgn}[\eta(t;k,\eta^o(i,k)) \cdot (c+bk_{n+1})] \tag{15}$$

Equations (14), (15) and (4) can be solved simultaneously to obtain $x(T;k^*,g^i)$ from which $C_i(k^*)$ is readily obtained.

Some simplification can be obtained by looking further at the explicit form of (12). Since $x(T;k^*,g^i)$ is the value at time T of the solution to (4), $C_i(k^*)$ may be written as

$$C_i(k^*) = d^i \cdot \int_0^T \exp[(A+bk')\,(T-t)]\,(c+bk_{n+1})\; g^i(t)dt \tag{16}$$

From (15) and the definitions of $\eta(t;k,\eta^{o})$ and $\eta^{o}(i,k)$ it is possible to write

$$g_i(t) = \text{sgn}\left\{\exp[-(A+bk')'t]\exp[(A+bk')'T]d^i \cdot (c+bk_{n+1})\right\}$$

$$= \text{sgn}\left\{\exp[(A+bk')'(T-t)]d^i \cdot (c+bk_{n+1})\right\}$$

Substituting this expression into (16) and taking note of the identity, $d^i \cdot \exp[(A+bk')(T-t)](c+bk_{n+1}) = \exp[(A+bk')'(T-t)]d^i \cdot (c+bk_{n+1})$ the result is

$$C_i(k^*) = \int_0^T |\exp[(A+bk')'(T-t)]d^i \cdot (c+bk_{n+1})|dt \tag{17}$$

or

$$C_i(k^*) = \int_0^T |\exp[(A+bk')(T-t)](c+bk_{n+1}) \cdot d^i|dt \tag{18}$$

By making the change of variable $s = T-t$, (17) and (18) may be written as

$$C_i(k^*) = \int_0^T |\exp[(A+bk')'s]d^i \cdot (c+bk_{n+1})|ds \tag{19}$$

$$C_i(k^*) = \int_0^T |\exp[(A+bk')s](c + bk_{n+1}) \cdot d^i|\, ds \tag{20}$$

The right hand sides of (19) and (20) can be computed by solving systems of differential equations. Let $v^i(s) = \exp[(A+bk')'s]d^i$ and $p(s) = \exp[(A+bk')s](c+bk_{n+1})$. Then

$C_i(k^*)$ could be found by solving one of the systems

$$\dot{v}^i = (A+bk')' v^i, \qquad v^i(0) = d^i$$
$$\dot{w}^i = |v^i \cdot (c+bk_{n+1})|, \quad w_i(0) = 0 \tag{21}$$

$$\dot{p} = (A+bk')p, \qquad p(0) = c+bk_{n+1}$$
$$\dot{q}_i = |p \cdot d^i|, \quad q_i(0) = 0 \tag{22}$$

If (21) is used, $C_i(k^*)$ as given by (19) is equal to $w_i(T)$. If (22) is used, $C_i(k^*)$ as given by (20) is equal to $q_i(T)$. To obtain $C(k^*)$ using (22) the system for $p(t)$ must be solved with an equation for $q_i(t)$ for each $i = 1,2,\ldots,m$ and then take the maximum of the m values $q_i(T)$. This involves, then, the solution of a system of n+m first order equations. The system of equations may be decomposed into a system of n linear homogeneous equations, and m equations which are each piece-wise linear of a rather simple type.

The method will now be applied to particular examples with the intention of demonstrating and clarifying the computations involved. As a by-product some indication of the characteristics of a controller which is optimal according to the criterion being considered is obtained.

EXAMPLES

The first two examples considered are second-order systems with m = 2. In these examples explicit formulas for $C_i(k^*)$ are obtained and the optimal controllers are determined when T is assumed to approach infinity.

The third example is a fourth order system which corresponds to the rigid body representation of a launch booster. A typical set of data and typical control configuration are assumed. The computational method for this example requires the aid of a computer and some of the computer results are presented.

EXAMPLE 1

The system considered in this example is represented in the form (1) by

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} u + \begin{vmatrix} 0 \\ 1 \end{vmatrix} g, \qquad \begin{matrix} x_1(0) = 0 \\ x_2(0) = 0 \end{matrix}$$

For the control criterion, d^1 and d^2 are chosen to be

$$d^1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \qquad d^2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}.$$

The class of controllers, U, is assumed to be

$$U = \left\{ u(x,g):\ u(x,g) = k_1 x_1 + k_2 x_2,\ k_1^2 + k_2^2 = 1 \right\}$$

i.e., $u(x,g)$ is an allowable controller if it is linear, fixed-gain with no disturbance feedback and the sum of the squares of the gains is one.

For convenience let $k_1 = \cos\theta$ and $k_2 = \sin\theta$. Then k is determined by θ, and since k^* is determined by k ($k_3 = 0$) dependence on k^* and k will be denoted by dependence on θ, with θ restricted by the inequality $0 \leq \theta < 2\pi$. Thus

$$A_\theta = A + bk' = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} \, |\cos\theta, \ \sin\theta|$$

$$= \begin{vmatrix} 0 & 1 \\ \cos\theta & \sin\theta \end{vmatrix}$$

Using the form of $C_i(\theta)$ given by (20) it is possible to write

$$C_i(\theta) = \int_0^T |p_i(t;\theta)|\,dt, \quad i = 1,2 \tag{23}$$

where $p(t;\theta)$ satisfies $\dot{p} = A_\theta p$, $p_1(0;\theta) = 0$, $p_2(0;\theta) = 1$ which are the first n equations in (22). Then $C_i(\theta)$ may be computed explicitly as follows. Let $\theta_1 = \cos^{-1}(2 - \sqrt{5})$, $\theta_2 = 2\pi - \theta_1$

$$\lambda_1(\theta) = \frac{1}{2}\,[\sin\theta + \sqrt{\sin^2\theta + 4\cos\theta}\ ], \text{ and}$$

$$\lambda_2(\theta) = \frac{1}{2}\,[\sin\theta - \sqrt{\sin^2\theta + 4\cos\theta}\ ].$$

Then $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are the characteristic roots of the matrix A_θ. For θ on the interval $[0,2\pi)$, $\lambda_1(\theta)$ and $\lambda_2(\theta)$ have the following properties:

For $0 \leq \theta < \theta_1$, $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are real and distinct.

For $\theta = \theta_1$, $\lambda_1(\theta) = \lambda_2(\theta) = \sqrt{\sqrt{5} - 2}$.

For $\theta_1 < \theta < \pi$, $\lambda_1(\theta) = \overline{\lambda_2(\theta)}$, $\mathrm{Re}[\lambda_1(\theta)] > 0$.

For $\theta = \pi$, $\lambda_1(\theta) = \overline{\lambda_2(\theta)}$, $\mathrm{Re}[\lambda_1(\theta)] = 0$.

For $\pi < \theta < \theta_2$, $\lambda_1(\theta) = \overline{\lambda_2(\theta)}$, $\mathrm{Re}[\lambda_2(\theta)] < 0$.

For $\theta = \theta_2$, $\lambda_1(\theta) = \lambda_2(\theta) = -\sqrt{\sqrt{5} - 2}$.

For $\theta_2 < \theta < 2\pi$, $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are real and distinct.

It may be easily verified that when $\lambda_1(\theta) \neq \lambda_2(\theta)$, then

$p_1(t;\theta) = [\exp(\lambda_1(\theta)t) - \exp(\lambda_2(\theta)t]/(\lambda_1-\lambda_2)$. If $\lambda_1(\theta) = \lambda_2(\theta) = \lambda(\theta) = \frac{1}{2}\sin\theta$, then $p_1(t;\theta) = t\exp(\lambda(\theta)t)$. The explicit form of $p_2(t;\theta)$ may be obtained from these expressions for $p_1(t;\theta)$ using the equation $p_2(t;\theta) = \dot{p}_1(t;\theta)$.

When $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are real and distinct, $p_1(t;\theta)$ and $p_2(t;\theta)$ are of constant sign for $t > 0$. Therefore, when $0 \leq \theta < \theta_1$ or $\theta_2 < \theta < 2\pi$ it is evident that

$$\int_0^T |p_i(t;\theta)|\,dt = \left|\int_0^T p_i(t;\theta)\,dt\right| \qquad \text{for } i = 1,2$$

Hence

$$C_1(\theta) = \begin{cases} |[\lambda_2(\theta)\exp(\lambda_1(\theta)T)-\lambda_1(\theta)\exp(\lambda_2(\theta)T)]/\lambda_1(\theta)\lambda_2(\theta)[\lambda_2(\theta)-\lambda_1(\theta)]|, & \text{if } \cos\theta\neq 0 \\ |\exp(\lambda_1(\theta)T) - 1 - T\,\lambda_1(\theta)|/[\lambda_1(\theta)]^2, & \cos\theta = 0 \end{cases}$$

$$C_2(\theta) = |\exp(\lambda_1(\theta)T) - \exp(\lambda_2(\theta)T)|/[\lambda_1(\theta) - \lambda_2(\theta)]$$

when $0 \leq \theta < \theta_1$ or $\theta_2 < \theta < 2\pi$.

For the special cases, $\theta = \pi/2, 3\pi/2, \theta_1, \theta_2$, the following results are obtained.

$$C_1(\pi/2) = e^T - 1 - T, \quad C_2(\pi/2) = e^T - 1$$

$$C_1(3\pi/2) = e^{-T} - 1 + T, \quad C_2(3\pi/2) = 1 - e^{-T}$$

$$C_1(\theta_1) = |1 - (1 - \lambda(\theta_1)T)\exp(\lambda(\theta_1)T)|/[\lambda(\theta_1)]^2$$

$$C_2(\theta_1) = T\exp(\lambda(\theta_1)T)$$

$$C_1(\theta_2) = |1 - (1 - \lambda(\theta_2)T)\exp(\lambda(\theta_2)T)|/[\lambda(\theta_2)]^2$$

$$C_2(\theta_2) = \begin{cases} T\exp(\lambda(\theta_2)T), & T \leq t_o = -1/\lambda(\theta_2) \\ -T\exp(\lambda(\theta_2)T) - 2/e\lambda(\theta_2), & T > t_o \end{cases}$$

When $\theta_1 < \theta < \theta_2$ the imaginary part of $\lambda_1(\theta)$ is not zero. Denoting the real and imaginary parts of $\lambda_1(\theta)$ by $\alpha(\theta)$ and $\beta(\theta)$ respectively, it is possible to obtain

$$C_1(\theta) = \frac{1}{\beta(\theta)} \int_0^T |\exp(\alpha(\theta)t)\ \sin(\beta(\theta)t)|\,dt \tag{24}$$

and

$$C_2(\theta) = \frac{1}{\beta(\theta)} \int_0^T \exp(\alpha(\theta)t)\,|\alpha(\theta)\sin(\beta(\theta)t)+\beta(\theta)\cos(\alpha(\theta)t)|\,dt \tag{25}$$

If T is allowed to approach infinity the following results are obtained. For $0 \leq \theta \leq \pi$ or $3\pi/2 \leq \theta < 2\pi$, $C(\theta)$ approaches infinity. For $\theta_2 \leq \theta < 3\pi/2$, $C(\theta) = |\sec\theta|$. For $\pi < \theta < \theta_2$, $C(\theta) = |\sec\theta| \coth[-\alpha(\theta)\pi/2\beta(\theta)]$. The minimum of $C(\theta)$ occurs at θ_o where $\pi < \theta_o < \theta_2$. The value of θ_o was determined to be approximately 230° and $C(230^\circ) = 2.24$. The damping ratio and frequency for the closed loop system determined by $u = x_1 \cos 230^\circ + x_2 \sin 230^\circ$ are approximately 0.477 and 0.802 respectively.

EXAMPLE 2

The system considered in this example is represented in the form (1) by

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} u + \begin{vmatrix} 0 \\ 1 \end{vmatrix} g, \qquad \begin{matrix} x_1(0) = 0 \\ x_2(0) = 0 \end{matrix}$$

The control criterion and the class of allowable controllers are chosen to be the same as given in example 1, and θ is introduced in the same manner.

The analysis is much the same as in the first example. In this case let

$$\lambda_1(\theta) = \frac{1}{2}\,(\sin\theta - 1 + \sqrt{(\sin\theta-1)^2 + 4\cos\theta})$$
$$\lambda_2(\theta) = \frac{1}{2}\,(\sin\theta - 1 - \sqrt{(\sin\theta-1)^2 + 4\cos\theta})$$

Also let $\theta_1 \approx 5\pi/4$ satisfy $(\sin\theta_1 - 1)^2 = -4\cos\theta_1$, and for $\pi/2 < \theta < \theta_1$ define $\alpha(\theta)$ and $\beta(\theta)$ by

$$\alpha(\theta) = \frac{1}{2}\,(\sin\theta - 1)$$
$$\beta(\theta) = \frac{1}{2}\sqrt{-4\cos\theta - (\sin\theta-1)^2}.$$

Then $C_1(\theta)$ may be expressed as: For $0 \leq \theta < \pi/2$, $\theta_1 < \theta < 3\pi/2$, $3\pi\ 2 < \theta < 2\pi$;

$$C_1(\theta) = |\sec\theta|\,\big|1 - [\lambda_1(\theta)\exp(\lambda_2(\theta)T) - \lambda_2(\theta)\exp(\lambda_1(\theta)T)]/[\lambda_1(\theta) - \lambda_2(\theta)]\big|$$

$$C_2(\theta) = |\exp(\lambda_1(\theta)T) - \exp(\lambda_2(\theta)T)/[\lambda_1(\theta) - \lambda_2(\theta)]|.$$

$$C_1(\pi/2) = T^2/2,\ C_2(\pi/2) = T$$

$$C_1(\theta_1) = |1 - (1 - \lambda(\theta_1)T)\exp(\lambda(\theta_1)T)|/[\lambda(\theta_1)]^2$$

$$C_2(\theta_1) = \begin{cases} T\exp(\lambda(\theta_1)T), & T \leq t_o = -1/\lambda(\theta_1) \\ -T\exp(\lambda(\theta_1)T) - 2/e\lambda(\theta_1), & T > t_o \end{cases}$$

$$C_1(3\pi/2) = |T - 1 + \exp(-2T)|/2$$

$$C_2(3\pi/2) = [1 - \exp(-2T)]/2$$

For $\pi/2 < \theta < \theta_1$, $C_1(\theta)$ and $C_2(\theta)$ are given by (24) and (25).

Again if T is allowed to approach infinity the value of θ, say θ_o, which minimizes $C(\theta)$ can be determined and is approximately

202°. The value of $C(202^{\circ})$ is approximately 1.17 and the damping ratio and frequency of the closed loop system with $\theta = 202^{\circ}$ are approximately 0.713 and .963 respectively.

EXAMPLE 3

This example is chosen to illustrate the techniques developed above for a more realistic approximation to a launch booster control problem. It is intended to be mainly illustrative, and the numerical results are presented as an indication of the characteristics of the optimal controller defined by the criterion expressed in (2).

A linear representation of the longitudinal equations of motion of a rigid vehicle is considered. The control configuration is chosen as a linear fixed gain system with pitch rate and lagged pitch attitude and normal acceleration feedbacks (Fig. 1). It is assumed that three gains, the time constant of the lag network and the accelerometer location are to be chosen satisfying certain constraints such that they are optimum according to a criterion of the form given in (2).

The assumed rigid body equations of motion are

$$\begin{aligned} \ddot{\phi} &= -c_1\,\alpha - c_2\,\beta \\ \ddot{z} &= \gamma_1\,\alpha + \gamma_2\,\phi + \gamma_3\,\beta \\ \alpha &= \phi + (v_{w} - \dot{z})/v \end{aligned} \tag{26}$$

The control equation is

$$\tau\dot{\beta}+\beta = K_3\left\{\dot{\phi}+K_2\,\phi + (K_1\gamma_1-c_1[\tau+K_1(C_M-C_G)]\,\alpha + (K_1\gamma_3-c_2[\tau+K_1(C_M-C_G)]\,\beta\right\}$$

Introducing $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = \dot{z}$, $x_4 = \beta$, the closed loop equations may be written as

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -c_1 & 0 & c_1/v & -c_2 \\ \gamma_1+\gamma_2 & 0 & -\gamma_1/v & \gamma_3 \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} + \begin{vmatrix} 0 \\ -c_1/v \\ \gamma_1/v \\ -k_3 \end{vmatrix} v_w \qquad (27)$$

where:

$$k_1 = (K_3/\tau)\ (K_2 + K_1\ \gamma_1 - c_1\ [\tau + K_1\ (C_M - C_G)])$$

$$k_2 = (K_3/\tau)$$

$$k_3 = (-K_3/\tau v)\ (K_1\ \gamma_1 - c_1\ [\tau + K_1\ (C_M - C_G)])$$

$$k_4 = [K_3\ (K_1\ \gamma_3 - c_2\ [\tau + K_1\ (C_M - C_G)]) - 1]/\tau$$

Taking the initial condition for (27) to be $x(0) = 0$, it may be seen that the equations are in the form given by (4). The following identification may be helpful: $k_5 = -k_3$, $u(x,g) = k_1\ x_1 + k_2\ x_2 + k_3\ x_3 + k_4\ x_4 + k_5\ g$, v_w plays the role of g, and

$$A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -c_1 & 0 & c_1/v & -c_2 \\ \gamma_1+\gamma_2 & 0 & -\gamma/v & \gamma_3 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad b = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} \quad c = \begin{vmatrix} 0 \\ -c_1/v \\ \gamma_1/v \\ 0 \end{vmatrix} \quad k = \begin{vmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{vmatrix}$$

It is clear from an inspection of the equation (27) that the system is determined by the four real parameters k_1, k_2, k_3, and k_4. These parameters may be thought of as pseudo-gains. The optimization problem can be stated in terms of these parameters if they are constrained so that the solutions (of the defining equations for k_1, k_2, k_3, k_4) for the gains, time constant and

accelerometer location are acceptable, i.e., the gains, time constant and accelerometer location are real and the accelerometer is located on the vehicle. To clarify this point and actually determine the constraints the following manipulations are performed. If it is assumed that $k_2 \neq 0$, (that is, that there is control) then the gains, K_1, K_2, K_3 and the time constant τ can be solved for in terms of k_1, k_2, k_3, k_4, and the accelerometer location, $C_M - C_G$, as follows

$$K_3 = k_2 \tau$$

$$K_2 = (k_1 + v\ k_3)/k_2$$

$$K_1 = [(v\ k_3/k_2) - c_1 \tau]/[-\gamma_1 + c_1(C_M - C_G)]$$

and τ is a solution of the quadratic equation

$$k_2(c_1\gamma_3 - c_2\gamma_1)\tau^2 + (k_4[-\gamma_1 + c_1(C_M - C_G)] - v\ k_3[\gamma_3 - c_2(C_M - C_G)])\tau + [c_1(C_M - C_G) - \gamma_1] = 0$$

In order for τ to be real it is necessary that

$$[(k_4\ c_1 + v\ k_3\ c_2)\ (C_M - C_G) - k_4\ \gamma_1 - v\ k_3\ \gamma_3]^2 - 4k_2(c_1\gamma_3 - c_2\gamma_1)[c_1(C_M - C_G) - \gamma_1]$$

is greater than zero.

Denoting by b_8 and b_9 the minimum and maximum values respectively of $C_M - C_G$ such that C_M represents an accelerometer location that is on the vehicle, it is possible to express the constraint on the k's in the form:

$$E(k) \cap [b_8, b_9] \neq \phi \tag{28}$$

where

$$E(k) = \left\{\lambda : [(k_4c_1+vk_3c_2)\lambda - k_4\gamma_1-vk_3\gamma_3]^2 \geq 4k_2(c_1\gamma_3-c_2\gamma_1)(c_1\lambda-\gamma_1)\right\}.$$

This constraint statement is just a mathematical way of stating that k must be chosen so that there is some accelerometer location on the vehicle for which the corresponding value of τ is real. It should be noted at this point that if it is desired to restrict the accelerometer location to some particular position or to be in some particular range of positions on the vehicle, then the interval, $[b_8, b_9]$, in (28) may be replaced by the appropriate point set which corresponds to the desired restriction of accelerometer locations.

Now the problem of optimizing the pseudo-gain vector k, (k^* being determined by k since $k_5 = -k_3$), subject to the constraint (28) according to a criterion of the form (2), namely

$$C(k) = \max_{1\leq i\leq 4} C_i(k); \; C_i(k) = \max_{0\leq t\leq T} \; \max_{v_w\in G} |d^i \cdot x(t;k,v_w)| \qquad (29)$$

is formulated.

To be precise, the weighting vector d^i, i = 1, 2, 3, 4, is taken to be

$$d^1 = \begin{vmatrix} r_1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad d^2 = \begin{vmatrix} 0 \\ r_2 \\ 0 \\ 0 \end{vmatrix} \qquad d^3 = \begin{vmatrix} 0 \\ 0 \\ r_3 \\ 0 \end{vmatrix} \qquad d^4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ r_4 \end{vmatrix} \quad .$$

Another restriction is imposed upon the components of the vector k. The intent of this restriction is to remove the possibility of infinite gains. An artificial restriction is imposed, but it is motivated by the following heuristic considerations.

First assume the restriction is of the form $\lambda_1^2 k_1^2 + \lambda_2^2 k_2^2 + \lambda_3^2 k_3^2 + \lambda_4^2 k_4^2 \leq \lambda_5^2$ with some real numbers λ_i, $i = 1,2,3,4,5$. To obtain some estimate of the desired values of the λ_i, it may be assumed that an inequality of the following type is to hold,

$$k_1^2 \, |x_1|^2_{max} + k_2^2 \, |x_2|^2_{max} + k_3^2 \, |x_3 - v_w|^2_{max} + k_4^2 |x_4|^2_{max} \leq \frac{1}{2} |\dot{x}_4|^2_{max} \quad (30)$$

This is motivated by the form of equation (27). The r_i, $i = 1,2,3,4$, are chosen to provide the desired ratios between $|x_1|_{max}$, $|x_2|_{max}$, $|x_3|_{max}$, $|x_4|_{max}$, i.e., so that $r_1 |x_1|_{max} = r_2 |x_2|_{max} = r_3 |x_3|_{max} = r_4 |x_4|_{max}$. Using these relations and introducing r_5 by an equation $r_5 \, |x_3 - v_w|_{max} = r_4 |x_4|_{max}$, (30) may be replaced by

$$2r_4^2 \, (k_1^2 \, r_1^{-2} + k_2^2 \, r_2^{-2} + k_3^2 \, r_5^{-2} + k_4^2 \, r_4^{-2}) \leq (|\dot{x}_4|_{max}/|x_4|_{max})^2 \quad (31)$$

The numbers in this expression could be chosen to specify the values of the λ_i's. In the numerical example a certain set of values of k was chosen to "span" a region of the type defined by (31).

The data used for this example represents that for a typical launch booster. Units for the data are meters, radians, and seconds. The numerical values are: $T = 30$, $c_1 = -.2165$,

$c_1/v = -.000427$, $c_2 = 1.1381$, $\gamma_1 + \gamma_2 = 27.66$, $\gamma_1/v = .0133$, $\gamma_3 = 17.65$, $b_8 = -28.5$, $b_9 = 75.4$, $r_1 = 1/2.35$, $r_2 = 1$, $r_3 = 1/57.3$, $r_4 = 1/1.33$.

A program for the H-800 digital computer was written to evaluate $C(k)$ by solving the system of equations (21) for this example. The first step in the program was to determine if (28) was satisfied and in fact determine allowable accelerometer locations. As a first cut at the problem the values for k_i were chosen from:

$$k_1 = 0, \pm\sqrt{2}(1.76)^{-1}, \pm\tfrac{1}{2}\sqrt{2}(1.76)^{-1}$$

$$k_2 = \pm\sqrt{2}(4/3), \pm\sqrt{2}(2/3)$$

$$k_3 = 0, \pm\sqrt{2}(2880)^{-1}, \pm\sqrt{2}(5760)^{-1}$$

$$k_4 = 0, \pm\sqrt{2}, \pm\tfrac{1}{2}\sqrt{2}$$

All possible combinations were taken subject to the restriction that at most one k_i was at its extreme value. This gave 216 cases. Of these 216 cases, $C(k)$ was computed for each k which satisfied (28). The computer time involved in this phase was approximately one hour and forty minutes. The minimizing allowable k, say k^1, from this set had components $k_1 = \frac{1}{2}\sqrt{2}(1.76)^{-1}$, $k_2 = 2\sqrt{2}/3$, $k_3 = -\sqrt{2}(2880)^{-1}$, $k_4 = -\frac{1}{2}\sqrt{2}$. The value of $C(k^1)$ was found to be approximately .0043. As a comparison the cost of no control, i.e., $k_2 = 0$, was computed to be 604.

As a second cut at the problem the following values for k_i were chosen: (representing a refinement of the grid about k^1)

k_1 = .20088261, .40176522, .60264783

k_2 = .47140452, .94280904, 1.41421356

k_3 = -.00049105, -.00036828

k_4 = -.35355339, -.70710678, -1.060066017

All possible combinations were used in this phase. The minimizing allowable k, say k^o, from this set had components: $k_1 = .60264783$, $k_2 = 1.41421356$, $k_3 = -.00049105$, $k_4 = -.70710678$. The value of $C(k^o)$ was found to be approximately .0033. The computation time for this phase was approximately twenty minutes. The characteristic roots of the closed loop system corresponding to k^o were computed to be approximately, -.0047, -.44, -.137 $\pm$ j 1.126. The allowable accelerometer locations corresponding to k^o were determined to be $-28.5 < C_M - C_G < 21.3$. With $C_M - C_G = -25$, two corresponding sets of values of τ, K_1, K_2, K_3 were computed to be .635 (.129), .029(.111), .250 (.250), and .900 (.182), respectively.

Transient responses for the extremal disturbances giving rise to $C_i(k^o)$ are shown in figures 2 - 5.

CONCLUSIONS

A method is developed for designing linear controllers for linear systems on the basis of minimizing disturbance effects. In general, the method requires the use of computers. However, it appears that the computation time is not excessive.

The method is applied to examples to illustrate the techniques involved. The numerical results obtained in the example corresponding to a rigid launch booster problem are indicative of

the nature of the given criterion.

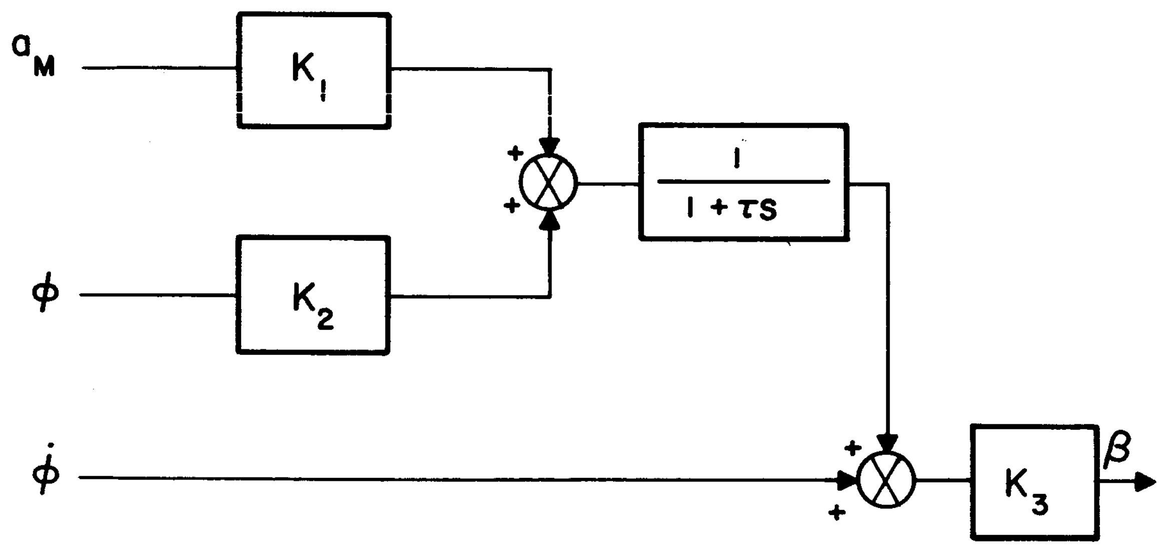
REFERENCE

1. Harvey, C. A., and E. B. Lee, "On the Uniqueness of Time-Optimal Control for Linear Processes", Journal of Math. Anal. and Appl., Vol. 5, No. 2, pp. 258 - 268, 1962.

LIST OF SYMBOLS

α	Angle of attack
β	Swivel motor deflection
γ_1	Aerodynamic force coefficient
γ_2	Vehicle longitudinal acceleration
γ_3	Control force coefficient
ϕ	Attitude angle
a_m	Local linear acceleration sensed by accelerometer
c_1	Specific aerodynamic restoring torque
c_2	Specific control torque
v	Magnitude of standard velocity of vehicle
v_w	Magnitude of wind velocity
z	Displacement of center of gravity of vehicle
C_G	Location of center of gravity
C_M	Location of accelerometer

Others defined in text.



$$a_M = (c_M - c_G)\ddot{\phi} + \gamma_1 \alpha + \gamma_3 \beta$$

Figure 1. Control Configuration for Example 3

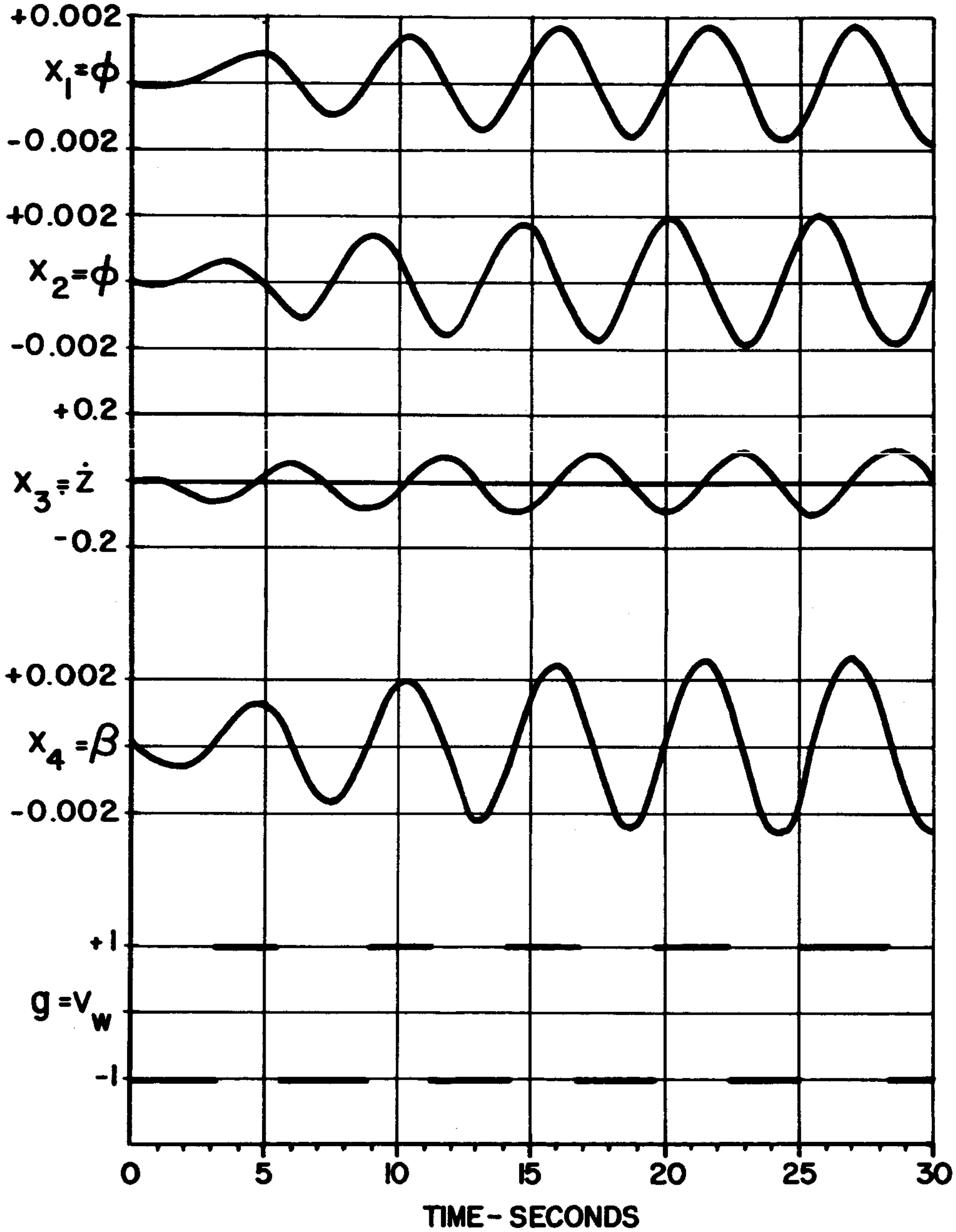


Figure 2. Response to Extremal Disturbance Which Maximizes x_1

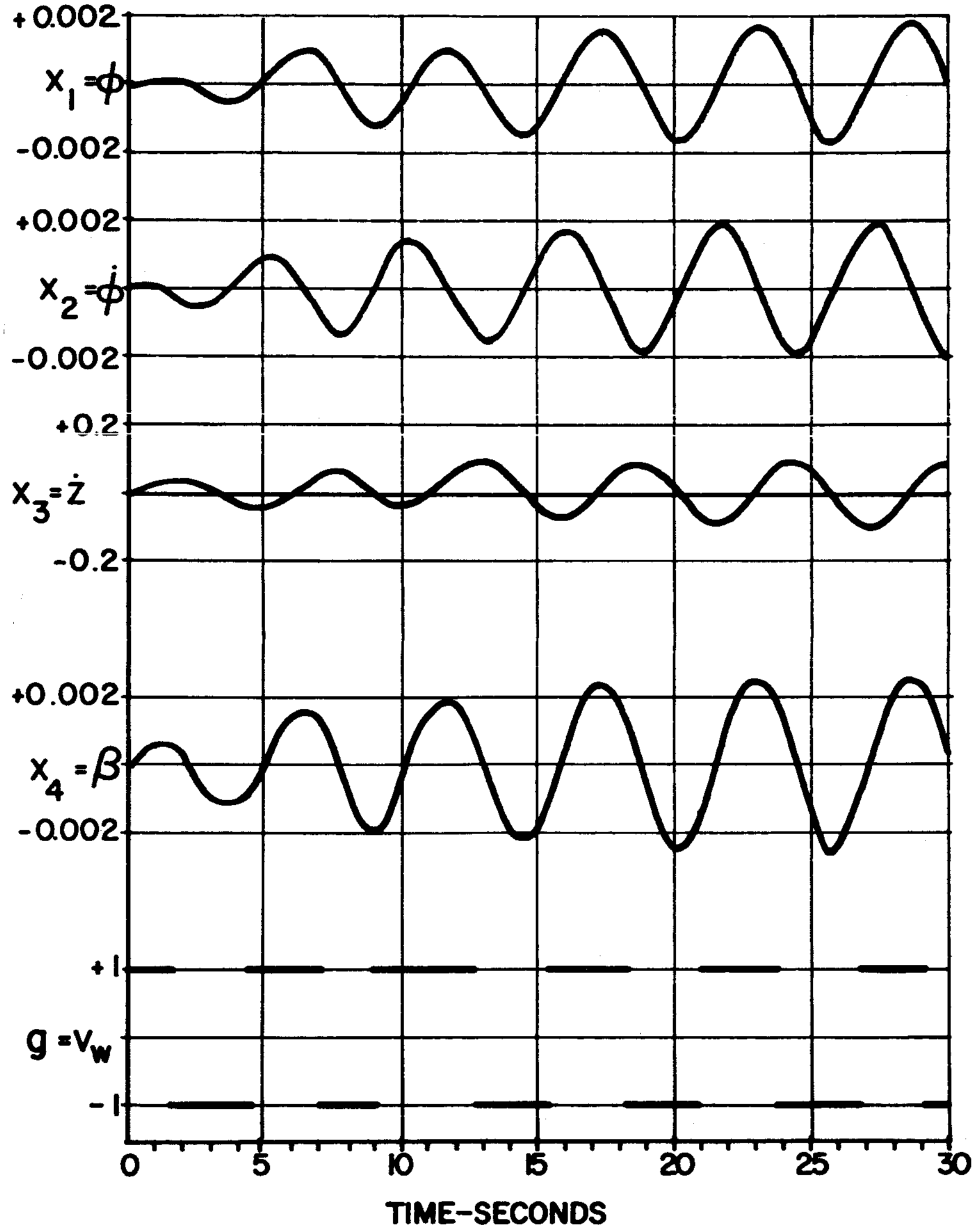


Figure 3. Response to Extremal Disturbance Which Maximizes x_2

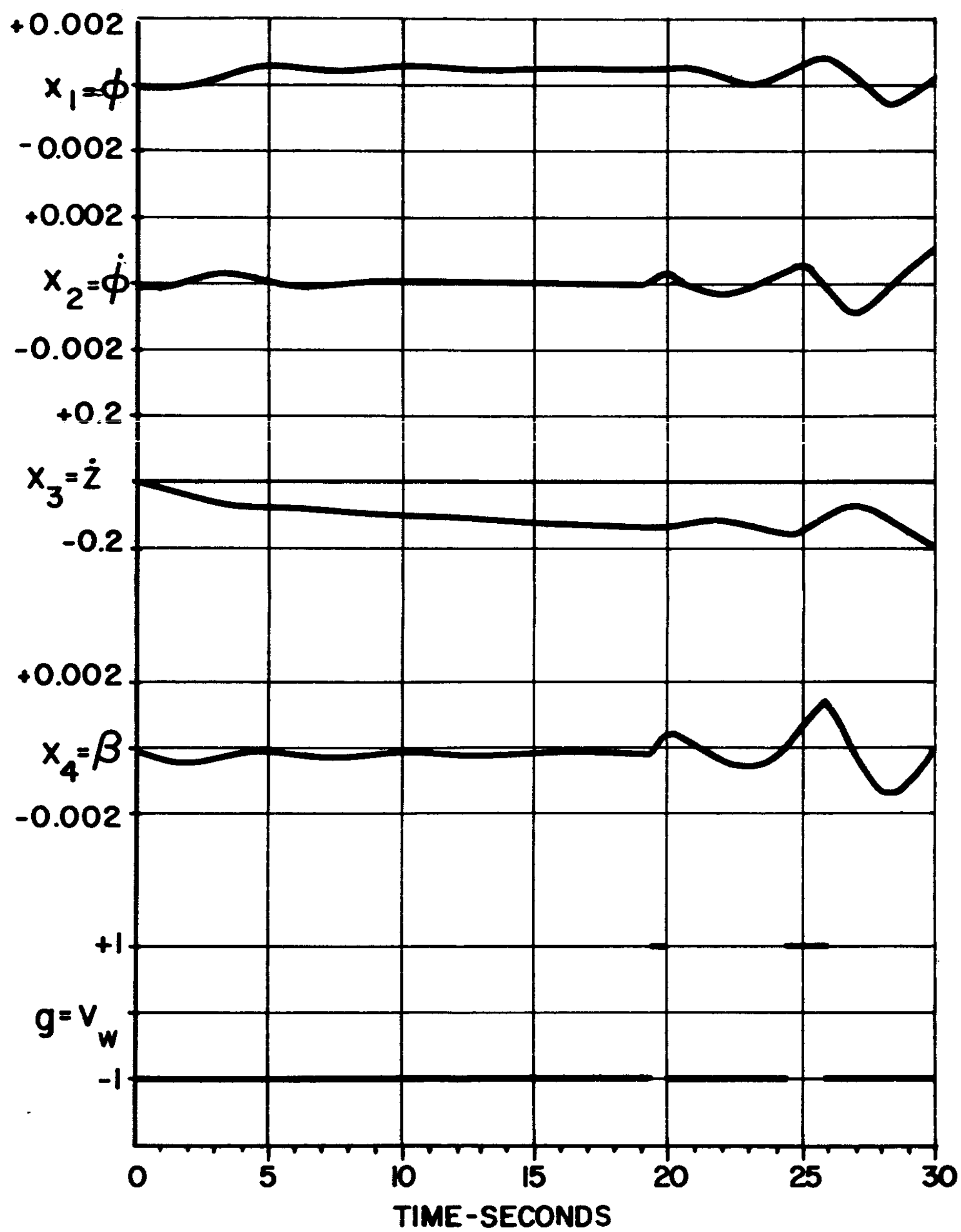


Figure 4. Response to Extremal Disturbance Which Maximizes x_3

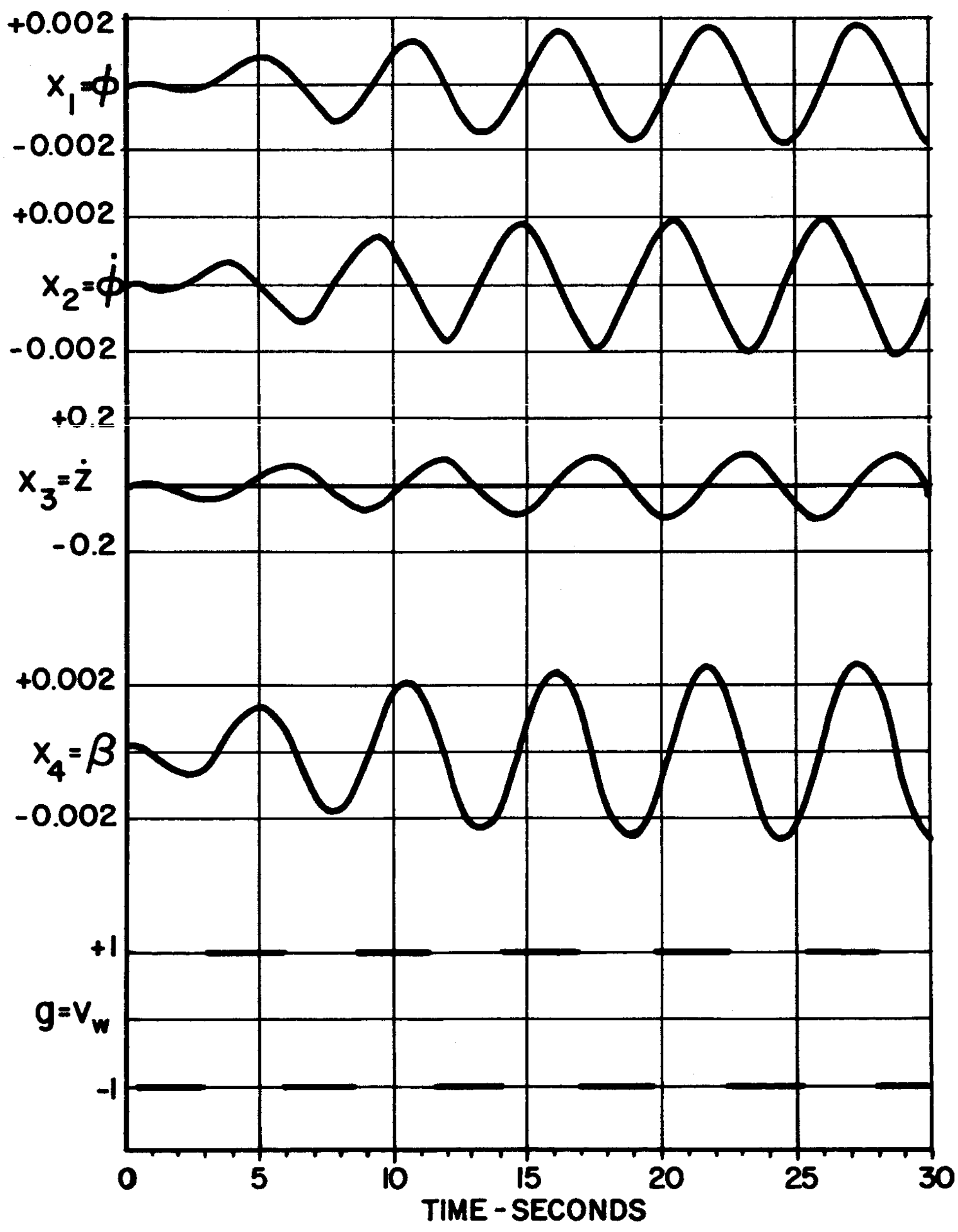


Figure 5. Response to Extremal Disturbance Which Maximizes x_4